SECTION 13.3: THE DOT PRODUCT

We've learned how add and subtract vectors and how to multiply vectors by scalars. In this section, we learn one way to multiply a vector times a vector.

DEFINITION: Given vectors $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$, the **dot product** of $\vec{v}$ and $\vec{w}$ is given by

$$\vec{v} \cdot \vec{w} = \langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle = v_1 w_1 + v_2 w_2$$

For example, if $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle 1, -2 \rangle$,then $\vec{v} \cdot \vec{w} = \langle 3, 4 \rangle \cdot \langle 1, -2 \rangle = (3)(1) + (4)(-2) = -5$.

The dot product extends to three dimensions as one would expect: if $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ then

$$\vec{v} \cdot \vec{w} = \langle v_1, v_2, v_3 \rangle \cdot \langle w_1, w_2, w_3 \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Note that the dot product takes two **vectors** and produces a **scalar**. For that reason, the quantity $\vec{v} \cdot \vec{w}$ is often called the **scalar product** of $\vec{v}$ and $\vec{w}$. The dot product enjoys the following properties.

PROPERTIES OF THE DOT PRODUCT

- **COMMUTATIVE PROPERTY:** For all vectors $\vec{v}$ and $\vec{w}$, $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
- **DISTRIBUTIVE PROPERTY:** For all vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.
- **SCALARS FLOAT:** For all vectors $\vec{v}$ and $\vec{w}$ and scalars k, $(k\vec{v}) \cdot \vec{w} = k(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (k\vec{w})$.
- **RELATION TO MAGNITUDE:** For all vectors $\vec{v}$, $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

Like most of the theorems involving vectors, the proof of these properties amounts to using the definition of the dot product and properties of real number arithmetic. We restrict our proofs to two dimensions since the three dimension proofs are shown similarly.

For example, to show the commutative property, let $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$. Then

$$\begin{array}{rcll} \vec{v} \cdot \vec{w} & = & \langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle & \\ & = & v_1 w_1 + v_2 w_2 & \text{Definition of Dot Product} \\ & = & w_1 v_1 + w_2 v_2 & \text{Commutativity of Real Number Multiplication} \\ & = & \langle w_1, w_2 \rangle \cdot \langle v_1, v_2 \rangle & \text{Definition of Dot Product} \\ & = & \vec{w} \cdot \vec{v} & \end{array}$$

For the scalar property, assume that $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ and k is a scalar. Then

$$\begin{array}{rcll} (k\vec{v}) \cdot \vec{w} & = & (k \langle v_1, v_2 \rangle) \cdot \langle w_1, w_2 \rangle & \\ & = & \langle kv_1, kv_2 \rangle \cdot \langle w_1, w_2 \rangle & \text{Definition of Scalar Multiplication} \\ & = & (kv_1)(w_1) + (kv_2)(w_2) & \text{Definition of Dot Product} \\ & = & k(v_1 w_1) + k(v_2 w_2) & \text{Associativity of Real Number Multiplication} \\ & = & k(v_1 w_1 + v_2 w_2) & \text{Distributive Law of Real Numbers} \\ & = & k \langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle & \text{Definition of Dot Product} \\ & = & k(\vec{v} \cdot \vec{w}) & \end{array}$$

For the last property, we note that if $\vec{v} = \langle v_1, v_2 \rangle$, then $\vec{v} \cdot \vec{v} = \langle v_1, v_2 \rangle \cdot \langle v_1, v_2 \rangle = v_1^2 + v_2^2 = \left(\sqrt{v_1^2 + v_2^2} \right)^2 = \|\vec{v}\|^2$.

EXAMPLE 1: Prove the identity: $\|\vec{v}-\vec{w}\|^2 = \|\vec{v}\|^2 - 2(\vec{v}\cdot\vec{w}) + \|\vec{w}\|^2$.

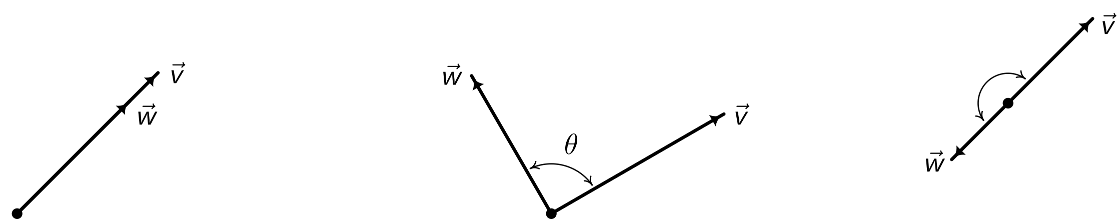
SOLUTION: We begin by rewriting $\|\vec{v}-\vec{w}\|^2$:

$$
\begin{aligned}
\|\vec{v}-\vec{w}\|^2 &= (\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w}) \\
&= (\vec{v}+[-\vec{w}])\cdot(\vec{v}+[-\vec{w}]) \\
&= (\vec{v}+[-\vec{w}])\cdot\vec{v} + (\vec{v}+[-\vec{w}])\cdot[-\vec{w}] \\
&= \vec{v}\cdot(\vec{v}+[-\vec{w}]) + [-\vec{w}]\cdot(\vec{v}+[-\vec{w}]) \\
&= \vec{v}\cdot\vec{v} + \vec{v}\cdot[-\vec{w}] + [-\vec{w}]\cdot\vec{v} + [-\vec{w}]\cdot[-\vec{w}] \\
&= \vec{v}\cdot\vec{v} + \vec{v}\cdot[(-1)\vec{w}] + [(-1)\vec{w}]\cdot\vec{v} + [(-1)\vec{w}]\cdot[(-1)\vec{w}] \\
&= \vec{v}\cdot\vec{v} + (-1)(\vec{v}\cdot\vec{w}) + (-1)(\vec{w}\cdot\vec{v}) + [(-1)(-1)](\vec{w}\cdot\vec{w}) \\
&= \vec{v}\cdot\vec{v} + (-1)(\vec{v}\cdot\vec{w}) + (-1)(\vec{v}\cdot\vec{w}) + \vec{w}\cdot\vec{w} \\
&= \vec{v}\cdot\vec{v} - 2(\vec{v}\cdot\vec{w}) + \vec{w}\cdot\vec{w} \\
&= \|\vec{v}\|^2 - 2(\vec{v}\cdot\vec{w}) + \|\vec{w}\|^2
\end{aligned}
$$

Hence, $\|\vec{v}-\vec{w}\|^2 = \|\vec{v}\|^2 - 2(\vec{v}\cdot\vec{w}) + \|\vec{w}\|^2$ as required.

Note in the midst of the previous calculations,[1] we see that $(\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w}) = \vec{v}\cdot\vec{v} - 2(\vec{v}\cdot\vec{w}) + \vec{w}\cdot\vec{w}$.

Suppose $\vec{v}$ and $\vec{w}$ are two nonzero vectors. If we draw $\vec{v}$ and $\vec{w}$ with the same initial point, we define the **angle between $\vec{v}$ and $\vec{w}$** to be the angle θ determined by the rays containing the vectors $\vec{v}$ and $\vec{w}$, as illustrated below. We require $0 \le \theta \le \pi$. (Think about why this is needed in the definition.)



$\theta = 0$ $\qquad$ $0 < \theta < \pi$ $\qquad$ $\theta = \pi$

THEOREM: GEOMETRY OF THE DOT PRODUCT: If $\vec{v}$ and $\vec{w}$ are nonzero vectors then

$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos(\theta),$$

where θ is the angle between $\vec{v}$ and $\vec{w}$.

We'll assume if $0 < \theta < \pi$, and leave the two extreme cases for you to think about. In this case, the vectors $\vec{v}$, $\vec{w}$ and $\vec{v}-\vec{w}$ determine a triangle with side lengths $\|\vec{v}\|$, $\|\vec{w}\|$ and $\|\vec{v}-\vec{w}\|$:

The Law of Cosines yields $\|\vec{v}-\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos(\theta)$.

[1] Does this look familiar? Why?

From the previous example, we also have that: $\|\vec{v}-\vec{w}\|^2 = \|\vec{v}\|^2 - 2(\vec{v}\cdot\vec{w}) + \|\vec{w}\|^2$.
Equating these two expressions for $\|\vec{v}-\vec{w}\|^2$ gives $\|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos(\theta) = \|\vec{v}\|^2 - 2(\vec{v}\cdot\vec{w}) + \|\vec{w}\|^2$ which reduces to $-2\|\vec{v}\|\|\vec{w}\|\cos(\theta) = -2(\vec{v}\cdot\vec{w})$. Hence, $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos(\theta)$, as required.

COROLLARY: Let $\vec{v}$ and $\vec{w}$ be nonzero vectors and let θ the angle between $\vec{v}$ and $\vec{w}$. Then

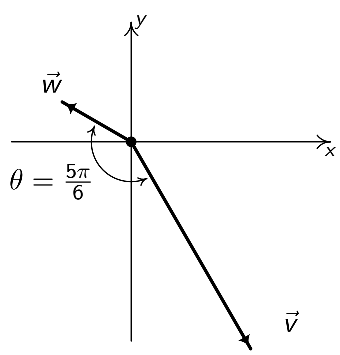
$$\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) = \cos^{-1}(\hat{v}\cdot\hat{w})$$

EXAMPLE 2: Find the angle between the following pairs of vectors. Graph each pair of vectors in standard position to check the reasonableness of your answer.

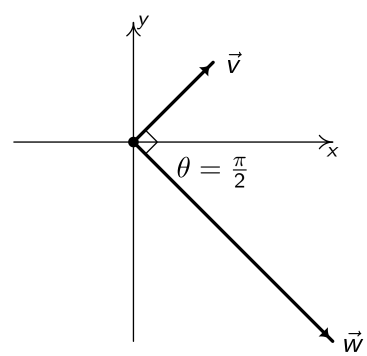
1. $\vec{v} = \left\langle 3, -3\sqrt{3}\right\rangle$, $\vec{w} = \left\langle -\sqrt{3}, 1\right\rangle$
2. $\vec{v} = \langle 2, 2\rangle$, $\vec{w} = \langle 5, -5\rangle$
3. $\vec{v} = \langle 3, -4\rangle$, $\vec{w} = \langle 2, 1\rangle$

SOLUTION: We use the formula $\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right)$:

1. We have $\vec{v}\cdot\vec{w} = \left\langle 3, -3\sqrt{3}\right\rangle \cdot \left\langle -\sqrt{3}, 1\right\rangle = -3\sqrt{3} - 3\sqrt{3} = -6\sqrt{3}$. Computing lengths of vectors, we find $\|\vec{v}\| = \sqrt{3^2 + (-3\sqrt{3})^2} = \sqrt{36} = 6$ and $\|\vec{w}\| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$. Hence, we find $\theta = \cos^{-1}\left(\frac{-6\sqrt{3}}{12}\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$. We check our answer geometrically by graphing this pair of vectors below on the left.
2. For $\vec{v} = \langle 2, 2\rangle$ and $\vec{w} = \langle 5, -5\rangle$, we find $\vec{v}\cdot\vec{w} = \langle 2, 2\rangle \cdot \langle 5, -5\rangle = 10 - 10 = 0$. Hence, it doesn't matter what $\|\vec{v}\|$ and $\|\vec{w}\|$ are, $\theta = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) = \cos^{-1}(0) = \frac{\pi}{2}$. We check our answer geometrically by graphing this pair of vectors below on the right.

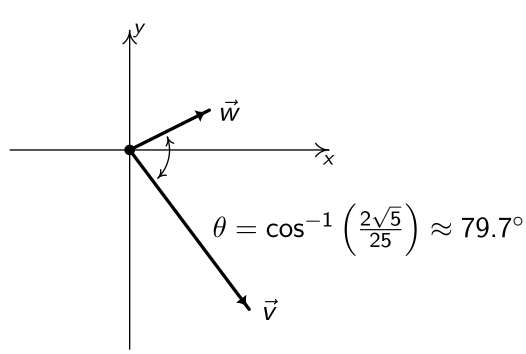


$\vec{v}$ and $\vec{w}$ from number 1



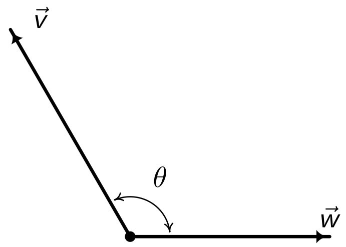
$\vec{v}$ and $\vec{w}$ from number 2

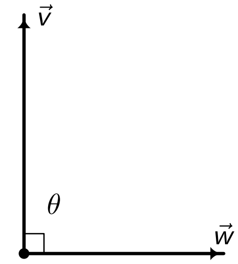
3. We find $\vec{v}\cdot\vec{w} = \langle 3, -4\rangle \cdot \langle 2, 1\rangle = 6 - 4 = 2$. Computing lengths, we find $\|\vec{v}\| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$ and $\vec{w} = \sqrt{2^2 + 1^2} = \sqrt{5}$, so $\theta = \cos^{-1}\left(\frac{2}{5\sqrt{5}}\right) = \cos^{-1}\left(\frac{2\sqrt{5}}{25}\right) \approx 79.7^\circ$.

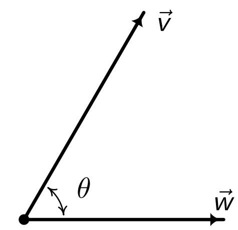


$\vec{v}$ and $\vec{w}$ from number 3

In general...







$\vec{v} \cdot \vec{w} < 0$	$\vec{v} \cdot \vec{w} = 0$	$\vec{v} \cdot \vec{w} > 0$
θ is obtuse	$\theta = 90°$, $\vec{v} \perp \vec{w}$	θ is acute

Of the three cases diagrammed above, the one which has the most mathematical significance moving forward is the orthogonal case. Hence, we state the corresponding theorem below.

THEOREM: THE DOT PRODUCT DETECTS ORTHOGONALITY

For nonzero vectors $\vec{v}$ and $\vec{w}$, $\vec{v} \perp \vec{w}$ if and only if $\vec{v} \cdot \vec{w} = 0$.

EXAMPLE 3: If $\vec{v} = \langle a, b \rangle$ is a nonzero vector, show $\vec{w}_1 = \langle b, -a \rangle$ and $\vec{w}_2 = \langle -b, a \rangle$ are orthogonal to $\vec{v}$.

How does this relate to the result that if a line has slope m, then the perpendicular line has slope $-\dfrac{1}{m}$?

DIRECTION COSINES: Re-framing the concept of $\hat{v}$.

If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\hat{\imath} =< 1, 0, 0 >$, then, on the one hand: $\vec{v} \cdot \hat{\imath} = \langle v_1, v_2, v_3 \rangle \cdot < 1, 0, 0 >= v_1$.
On the other hand, $\vec{v} \cdot \hat{\imath} = \|\vec{v}\| \, \|\hat{\imath}\| \cos(\alpha) = \|\vec{v}\| \cos(\alpha)$ since $\|\hat{\imath}\| = 1$ where α is the angle between $\vec{v}$ and $\hat{\imath}$.
Hence, $v_1 = \|\vec{v}\| \cos(\alpha)$ where α can be thought of as the angle between $\vec{v}$ and the positive x-axis.
Repeating this process with $\hat{\jmath}$ and $\hat{k}$, we get the following:

$$\vec{v} = \langle v_1, v_2, v_3 \rangle = \langle \|\vec{v}\| \cos(\alpha), \|\vec{v}\| \cos(\beta), \|\vec{v}\| \cos(\gamma) \rangle = \|\vec{v}\| \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$$

where α, β, and γ are the angles between $\vec{v}$ and the positive x-, y-, and z-axes respectively.

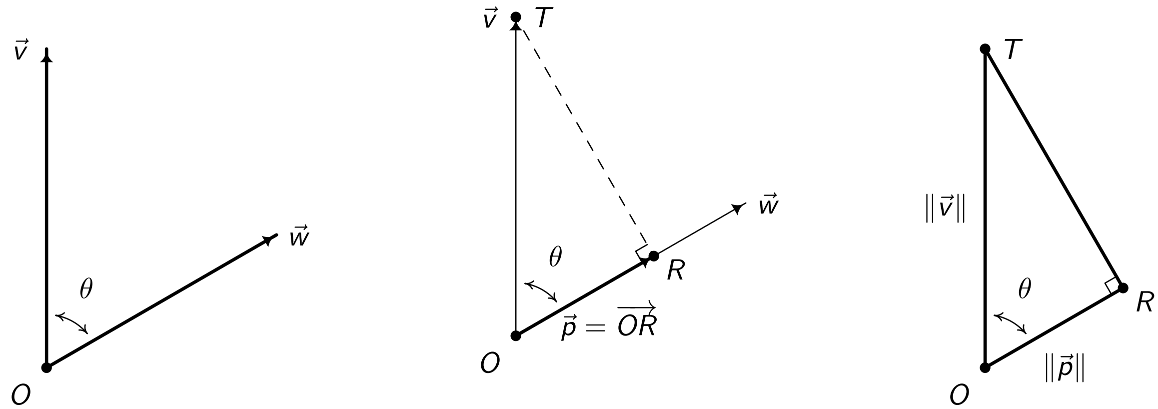
If $\vec{v} \neq 0$, then $\hat{v} = \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$. Hence, $\cos(\alpha)$, $\cos(\beta)$ and $\cos(\gamma)$ are the 'direction cosines' of $\vec{v}$.

EXAMPLE 4: Find the direction cosines of $\vec{v} = \langle 0, -1, 2 \rangle$.

QUESTION: How does this relate to the 'polar form' of a 2-D vector: $\vec{v} = \|\vec{v}\| < \cos(\theta), \sin(\theta) >$?

VECTOR PROJECTIONS:

Consider the two nonzero vectors $\vec{v}$ and $\vec{w}$ drawn with a common initial point O below. For the moment, assume that the angle between $\vec{v}$ and $\vec{w}$, θ, is acute. We wish to develop a formula for the vector $\vec{p}$, indicated below, which is called the **orthogonal projection of** $\vec{v}$ **onto** $\vec{w}$. The vector $\vec{p}$ is obtained geometrically as follows: drop a perpendicular from the terminal point T of $\vec{v}$ to the vector $\vec{w}$ and call the point of intersection R. The vector $\vec{p}$ is then defined as $\vec{p} = \overrightarrow{OR}$.



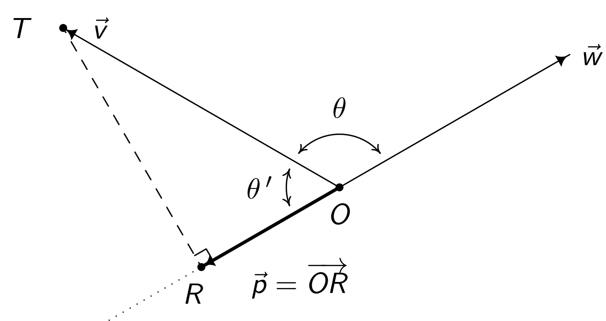
Like any vector, $\vec{p}$ is determined by its magnitude $\|\vec{p}\|$ and its direction $\hat{p}$ according to the formula $\vec{p} = \|\vec{p}\|\hat{p}$. Since we want $\hat{p}$ to have the same direction as $\vec{w}$, we have $\hat{p} = \hat{w}$.

Using right triangle trigonometry on $\triangle ORT$. We find $\cos(\theta) = \frac{\|\vec{p}\|}{\|\vec{v}\|}$, or, equivalently, $\|\vec{p}\| = \|\vec{v}\|\cos(\theta)$. Hence:

$$\|\vec{p}\| = \|\vec{v}\|\cos(\theta) = \frac{\|\vec{v}\|\|\vec{w}\|\cos(\theta)}{\|\vec{w}\|} = \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|} = \vec{v}\cdot\left(\frac{1}{\|\vec{w}\|}\vec{w}\right) = \vec{v}\cdot\hat{w}.$$

Hence, $\|\vec{p}\| = \vec{v}\cdot\hat{w}$, and since $\hat{p} = \hat{w}$, we have $\vec{p} = \|\vec{p}\|\hat{p} = (\vec{v}\cdot\hat{w})\hat{w}$.

Now suppose that the angle θ between $\vec{v}$ and $\vec{w}$ is obtuse, and consider the diagram below.



In this case, we see that $\hat{p} = -\hat{w}$ and using the triangle $\triangle ORT$, we find $\|\vec{p}\| = \|\vec{v}\|\cos(\theta')$. Since $\theta + \theta' = \pi$, it follows that $\cos(\theta') = -\cos(\theta)$, which means $\|\vec{p}\| = \|\vec{v}\|\cos(\theta') = -\|\vec{v}\|\cos(\theta)$.

Rewriting this last equation in terms of $\vec{v}$ and $\vec{w}$ as before, we get $\|\vec{p}\| = -(\vec{v}\cdot\hat{w})$. Putting this together with $\hat{p} = -\hat{w}$, we get $\vec{p} = \|\vec{p}\|\hat{p} = -(\vec{v}\cdot\hat{w})(-\hat{w}) = (\vec{v}\cdot\hat{w})\hat{w}$ in this case as well.

If the angle between $\vec{v}$ and $\vec{w}$ is $\frac{\pi}{2}$ then it is easy to show that $\vec{p} = \vec{0}$. Since $\vec{v}\perp\vec{w}$ in this case, $\vec{v}\cdot\vec{w} = 0$. It follows that $\vec{v}\cdot\hat{w} = 0$ and $\vec{p} = \vec{0} = 0\hat{w} = (\vec{v}\cdot\hat{w})\hat{w}$ in this case, too. We have motivated the following.

DEFINITION: Let $\vec{v}$ and $\vec{w}$ be nonzero vectors.

1. The **(orthogonal) scalar projection of $\vec{v}$ onto $\vec{w}$**, denoted $\text{comp}_{\vec{w}}(\vec{v})$ is given by $\text{comp}_{\vec{w}}(\vec{v}) = \vec{v} \cdot \hat{w}$
   NOTE: $\vec{v} \cdot \hat{w}$ is how much of $\vec{v}$ is in $\vec{w}$'s direction.
2. The **(orthogonal) vector projection of $\vec{v}$ onto $\vec{w}$**, denoted $\text{proj}_{\vec{w}}(\vec{v})$ is given by $\text{proj}_{\vec{w}}(\vec{v}) = (\vec{v} \cdot \hat{w})\,\hat{w}$.
   NOTE: $\text{proj}_{\vec{w}}(\vec{v}) = \text{comp}_{\vec{w}}(\vec{v})\,\hat{w}$.

THEOREM: Alternate Formulas for Vector Projections: If $\vec{v}$ and $\vec{w}$ are nonzero vectors then

$$\text{comp}_{\vec{w}}(\vec{v}) = \vec{v} \cdot \hat{w} = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|}$$

$$\text{proj}_{\vec{w}}(\vec{v}) = (\vec{v} \cdot \hat{w})\,\hat{w} = \left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\right)\vec{w} = \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right)\vec{w}$$

EXAMPLE 5: Let $\vec{v} = \langle 1, 8 \rangle$ and $\vec{w} = \langle -1, 2 \rangle$. Find $\vec{p} = \text{proj}_{\vec{w}}(\vec{v})$. Check your answer geometrically.

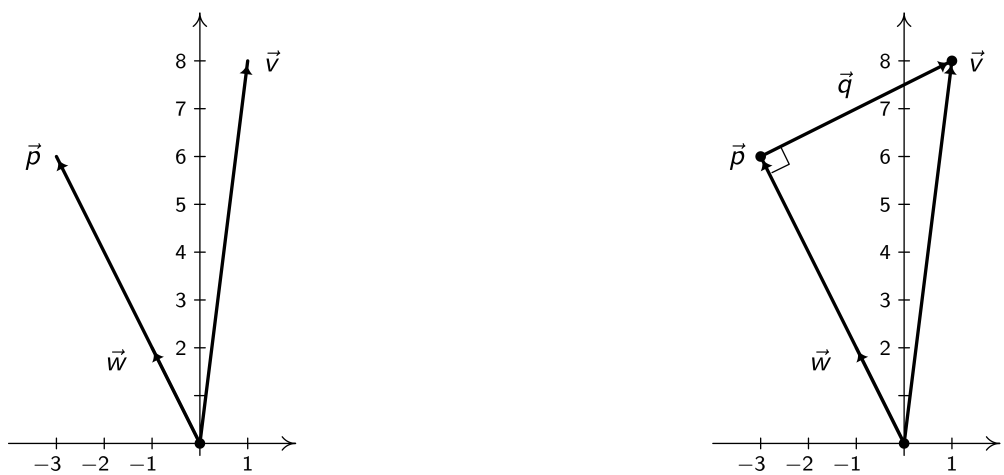
SOLUTION: We find $\vec{v} \cdot \vec{w} = \langle 1, 8 \rangle \cdot \langle -1, 2 \rangle = (-1) + 16 = 15$ and $\vec{w} \cdot \vec{w} = \langle -1, 2 \rangle \cdot \langle -1, 2 \rangle = 1 + 4 = 5$. Hence,

$$\vec{p} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\vec{w} = \frac{15}{5}\langle -1, 2 \rangle = \langle -3, 6 \rangle .$$

We plot $\vec{v}$, $\vec{w}$ and $\vec{p}$ in standard position below on the left. We see $\vec{p}$ has the same direction as $\vec{w}$, but we need to do more to show $\vec{p}$ in is indeed the **orthogonal** projection of $\vec{v}$ onto $\vec{w}$.

Consider the vector $\vec{q}$ whose initial point is the terminal point of $\vec{p}$ and whose terminal point is the terminal point of $\vec{v}$. From the definition of vector arithmetic, $\vec{p} + \vec{q} = \vec{v}$, so that $\vec{q} = \vec{v} - \vec{p}$.

Since $\vec{v} = \langle 1, 8 \rangle$ and $\vec{p} = \langle -3, 6 \rangle$, $\vec{q} = \langle 1, 8 \rangle - \langle -3, 6 \rangle = \langle 4, 2 \rangle$. To prove $\vec{q} \perp \vec{w}$, we compute the dot product: $\vec{q} \cdot \vec{w} = \langle 4, 2 \rangle \cdot \langle -1, 2 \rangle = (-4) + 4 = 0$. Hence, $\vec{q} \perp \vec{w}$ which completes our check.



VECTOR DECOMPOSITION THEOREM: If $\vec{v}$ and $\vec{w}$ are nonzero vectors, there are unique vectors $\vec{p}$ and $\vec{q}$ such that $\vec{v} = \vec{p} + \vec{q}$ where $\vec{p}$ is parallel to $\vec{w}$ and $\vec{q} \perp \vec{w}$.

We take $\vec{p} = \text{proj}_{\vec{w}}(\vec{v})$ and $\vec{q} = \vec{v} - \vec{p}$. Then $\vec{p}$ is, by definition, a scalar multiple of $\vec{w}$. Next, we compute $\vec{q} \cdot \vec{w}$.

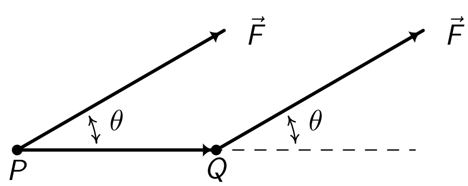
$$\begin{aligned}
\vec{q} \cdot \vec{w} &= (\vec{v} - \vec{p}) \cdot \vec{w} && \text{Definition of } \vec{q}. \\
&= \vec{v} \cdot \vec{w} - \vec{p} \cdot \vec{w} && \text{Properties of Dot Product} \\
&= \vec{v} \cdot \vec{w} - \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w} \right) \cdot \vec{w} && \text{Since } \vec{p} = \text{proj}_{\vec{w}}(\vec{v}). \\
&= \vec{v} \cdot \vec{w} - \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) (\vec{w} \cdot \vec{w}) && \text{Properties of Dot Product.} \\
&= \vec{v} \cdot \vec{w} - \vec{v} \cdot \vec{w} \\
&= 0.
\end{aligned}$$

Hence, $\vec{q} \cdot \vec{w} = 0$, as required. At this point, we have shown that the vectors $\vec{p}$ and $\vec{q}$ guaranteed by the theorem **exist**. Now we need to show that they are **unique** - that is, there is only **one** such way to decompose $\vec{v}$.

Suppose $\vec{v} = \vec{p} + \vec{q} = \vec{p}\,' + \vec{q}\,'$ where the vectors $\vec{p}\,'$ and $\vec{q}\,'$ satisfy the same properties as $\vec{p}$ and $\vec{q}$. Then $\vec{p} - \vec{p}\,' = \vec{q}\,' - \vec{q}$, so $\vec{w} \cdot (\vec{p} - \vec{p}\,') = \vec{w} \cdot (\vec{q}\,' - \vec{q}) = \vec{w} \cdot \vec{q}\,' - \vec{w} \cdot \vec{q} = 0 - 0 = 0$. The long and short of this computation is that $\vec{w} \cdot (\vec{p} - \vec{p}\,') = 0$.

Since $\vec{p}$ and $\vec{p}\,'$ are parallel to $\vec{w}$, there are scalars k and k' so that $\vec{p} = k\vec{w}$ and $\vec{p}\,' = k'\vec{w}$. This means $\vec{w} \cdot (\vec{p} - \vec{p}\,') = \vec{w} \cdot (k\vec{w} - k'\vec{w}) = \vec{w} \cdot ([k - k']\vec{w}) = (k - k')(\vec{w} \cdot \vec{w}) = (k - k')\|\vec{w}\|^2$.
Since $\vec{w} \neq \vec{0}$, $\|\vec{w}\|^2 \neq 0$, which means the only way $\vec{w} \cdot (\vec{p} - \vec{p}\,') = (k - k')\|\vec{w}\|^2 = 0$ is for $k - k' = 0$, or $k = k'$. This means $\vec{p} = k\vec{w} = k'\vec{w} = \vec{p}\,'$. Since $\vec{q}\,' - \vec{q} = \vec{p} - \vec{p}\,' = \vec{p} - \vec{p} = \vec{0}$, it must be that $\vec{q}\,' = \vec{q}$ as well.

WORK: In Physics, if a constant force F is exerted over a distance d, the **work** W done by the force is given by $W = Fd$. Here, the assumption is that the force is being applied in the direction of the motion. If the force applied is not in the direction of the motion, we can use the dot product to find the work done.
Consider the scenario sketched below in which the constant force $\vec{F}$ is applied to move an object from the point P to the point Q. Here the force is being applied at an angle θ as opposed to being applied directly in the direction of the motion.



To find the work W done in this scenario, we need to find how much of the force $\vec{F}$ is in the direction of the motion $\overrightarrow{PQ}$. This is precisely what the dot product $\vec{F} \cdot \widehat{PQ}$ represents.
Since the distance the object travels is $\|\overrightarrow{PQ}\|$, we get $W = (\vec{F} \cdot \widehat{PQ})\|\overrightarrow{PQ}\|$. Since $\overrightarrow{PQ} = \|\overrightarrow{PQ}\|\widehat{PQ}$, we can simplify this formula as follows: $W = (\vec{F} \cdot \widehat{PQ})\|\overrightarrow{PQ}\| = \vec{F} \cdot (\|\overrightarrow{PQ}\|\widehat{PQ}) = \vec{F} \cdot \overrightarrow{PQ}$.
We can rewrite $W = \vec{F} \cdot \overrightarrow{PQ} = \|\vec{F}\|\|\overrightarrow{PQ}\| \cos(\theta)$, where θ is the angle between the applied force $\vec{F}$ and the trajectory of the motion $\overrightarrow{PQ}$. We have proved the following.

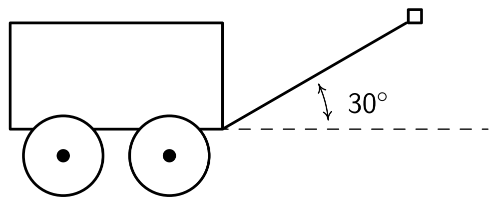
THEOREM: WORK AS A DOT PRODUCT:

If a constant force $\vec{F}$ is applied along the vector $\overrightarrow{PQ}$, the work W done by $\vec{F}$ is:

$$W = \vec{F} \cdot \overrightarrow{PQ} = \|\vec{F}\|\|\overrightarrow{PQ}\| \cos(\theta),$$

where θ is the angle between $\vec{F}$ and $\overrightarrow{PQ}$.

EXAMPLE 6: Taylor exerts a force of 10 pounds to pull her wagon a distance of 50 feet over level ground. If the handle of the wagon makes a 30° angle with the horizontal, how much work did Taylor do pulling the wagon? Assume the force of 10 pounds is exerted at a 30° angle for the duration of the 50 feet.



SOLUTION:

- **METHOD ONE:** One way is to find the vectors $\vec{F}$ and $\overrightarrow{PQ}$ and compute $W = \vec{F} \cdot \overrightarrow{PQ}$.

  To do this, we assume the origin is at the point where the handle of the wagon meets the wagon and the positive x-axis lies along the dashed line in the figure above.

  To find the force vector $\vec{F}$, we note the force in this situation is a constant 10 pounds, so $\|\vec{F}\| = 10$. Moreover, the force is being applied at a constant angle of $\theta = 30^\circ$ with respect to the positive x-axis. We get $\vec{F} = \|\vec{F}\| \langle \cos(\theta), \sin(\theta) \rangle = 10 \langle \cos(30^\circ, \sin(30^\circ)\rangle = \left\langle 5\sqrt{3}, 5 \right\rangle$.

  Since the wagon is being pulled along 50 feet in the positive x-direction, we find the displacement vector is $\overrightarrow{PQ} = 50\,\hat{\imath} = 50\,\langle 1, 0\rangle = \langle 50, 0\rangle$.

  Hence, $W = \vec{F} \cdot \overrightarrow{PQ} = \left\langle 5\sqrt{3}, 5 \right\rangle \cdot \langle 50, 0\rangle = 250\sqrt{3}$. Since force is measured in pounds and distance is measured in feet, we get $W = 250\sqrt{3}$ foot-pounds.

- **METHOD TWO:** Alternatively, we can use the formula $W = \|\vec{F}\| \|\overrightarrow{PQ}\| \cos(\theta)$. With $\|\vec{F}\| = 10$ pounds, $\|\overrightarrow{PQ}\| = 50$ feet and $\theta = 30^\circ$, we get $W = (10\,\text{pounds})(50\,\text{feet})\cos(30^\circ) = 250\sqrt{3}$ foot-pounds of work.

HOMEWORK: Section 13.3: 13 - 69 every other odd; 75*, 77*, 83, 87*